Spreading Metrics and Optimal Linear Arrangement

OLA: Given a graph $G(V,E)$ with edge weights $w_e \geq 0$

Find a bijection $f: V \to \{1,2,\dots n\}$ such that

$$\sum_{e=(u,v)} w_e |f(u)-f(v)| \text{ is minimized}$$

Map a graph vertices to a straight line without distorting the edge lengths too much.

$$d_{uv} = |f(u)-f(v)|$$

Spreading Metric Constraint

$$\min \sum_{e=(u,v)} w_e d_{uv} \quad \text{s.t.}$$

$$\forall S \subseteq V \text{ and } u \notin S, \quad \sum_{v \in S} d_{uv} \geq \frac{1}{4}|S|^2$$

makes d_{uv} a metric

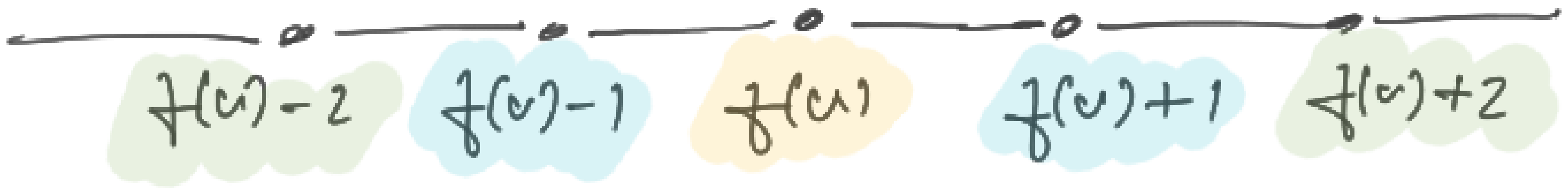
$$\begin{cases} d_{uv} = d_{vu} & \forall u,v \in V \\ d_{uv} \leq d_{uw} + d_{wv} & \forall u,v,w \in V \\ d_{uv} \geq 1 & \forall u \neq v \\ d_{uu} = 0 & \forall u \end{cases}$$

Above constraints make it a Spreading Metric

Claim: Above LP is a relaxation of OLA

- Given a mapping $f : V \to \{1 \dots n\}$, $d_{uv} = |f(u) - f(v)|$
- Consider $S \subseteq V$ and $u \notin S$
    - at most 2 vertices in S at dist $\underline{1}$ from u
    - at most 2 vertices in S at dist 2 from u

      $\vdots$
    - at most 2 vertices at dist $|S|/2$ from u



$$\therefore \sum_{v \in S} d_{uv} \geqslant 2\left(1 + 2 + \dots + |S|/2\right) \geqslant \frac{|S|^2}{4}$$

Hence a solⁿ to OLA is feasible for this LP

- Solving this LP: use Ellipsoid method with a separation oracle for the Spreading-Metric constraints.
  - Given a candidate solⁿ $\{d_{uv}\}$, need to check if $\exists\, S \subseteq V$ and $u \notin S$ s.t. $\sum_{v \in S} d_{uv} < |S|^2/4$
  - For each u, sort $V - \{u\}$ by distance from u
    $v_1 \quad v_2 \quad \cdots \quad v_{n-1}$
  - <u>Claim:</u> If $\exists\, S \not\ni u$ s.t. $\sum_{v \in S} d_{uv} < |S|^2/4$
    then $\sum_{i=1}^{|S|} d_{uv_i} < |S|^2/4$

    Because $\{v_1 \cdots v_{|S|}\}$ and S have same size and distances to S can only be greater
- Hence enough to test $\{v_1\}, \{v_1, v_2\}, \ldots, \{v_1\, v_2 \cdots v_{n-1}\}$

Obs: If d is a Spreading Metric, for any $S \subseteq V$ and $u \notin S$

$\exists\ v \in S$ st $d_{uv} \geq |S|/4$

- $\sum_{v \in S} d_{uv} \geq |S|^2/4 \Rightarrow$ avg dist to $u \geq |S|/4$

Approx Algo:

- Solve LP to get a Spreading Metric (V, d)
- Construct a Tree Metric (V', T) where
  - $V \subseteq V'$ is at the leaves of T
  - $\sum_{u,v \in V} w_{uv} T_{uv} \leq O(\log n) \sum_{u,v \in V} w_{uv} d_{uv}$
  - If $z \in V'$ is in the smallest subtree of T containing $u, v \in V$, then $T_{uv} \geq T_{uz}$
- Map V, the leaves of T, to $\{1, 2, \ldots n\}$ from Left to Right

$\hookrightarrow$ given f

Thm: This is a $O(\log n)$ approx to OLA

Pf: - Let $f: V \to \{1 \cdots n\}$ be the computed mapping

- For $e=(u,v)$ consider the smallest subtree of T containing u and v
  - leaves of this subtree are mapped to consecutive integers in a range $[a,b] \Rightarrow b-a+1$ leaves
  - $|f(u)-f(v)| \leq b-a$
  - $S =$ leaves except u, $|S| = b-a$
  - By Obs, $\exists z \in S$ st $d_{uz} \geq \frac{b-a}{4}$
  - By Tree Metric properties, $T_{uv} \geq T_{uz} \geq d_{uz} \geq \frac{b-a}{4}$
  - $\therefore |f(u)-f(v)| \leq 4T_{uv}$

$$\therefore \sum_{e=(u,v)} w_e |f(u)-f(v)| \leq \sum 4 w_e T_{uv}$$ (from above)

$$\leq O(\log n) \sum w_e d_{uv}$$ (by Tree Metric proportion)

$$\leq O(\log n)\, OPT_{OLA}$$ (by LP relaxation)

Cost of the approx sol^n

$\underline{\text{Th}^m}$: Given metric (V,d) and costs c_{uv} $\forall\, u,v \in V$, in deterministic polynomial time we can compute a tree metric (V', T) st

- $V \subseteq V'$ are leaves of T
- $T_{uv} \geq d_{uv}$ $\forall\, u,v \in V$
- $\sum_{u,v} c_{uv} T_{uv} \leq O(\log n) \sum_{u,v} c_{uv} d_{uv}$
- T is rooted and for any $z \in V$ that belongs to the smallest subtree of T containing $u,v \in V$ we have $T_{uv} \geq T_{uz}$

$\forall\, u,v$

$\mathbb{E}[T_{uv}] \leq O(\log n)\, d_{uv}$